The University of Alabama

Thirty-second Annual

High School Mathematics Tournament

Team Competition

November 16, 2013

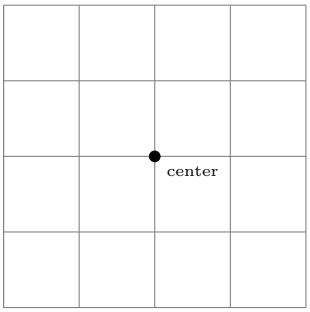
1. Do **not** turn this page until the proctor indicates that it is time to begin.
2. The notation $(f \circ g)(x)$ refers to composition of functions: $(f \circ g)(x) = f(g(x))$.
3. Throughout the test, the letter i represents the imaginary unit $i = \sqrt{-1}$, $\log(x)$ means $\log_{10}(x)$, and $\ln(x)$ means $\log_e(x)$.
4. All answers must be exact, unless specifically asked to do otherwise. Leave π, e, and radicals in the answer.
5. The test is 45 minutes in length. If you must leave the room, you **MAY NOT** re-enter the room before time is called.
6. Answers to the questions must be entered on the correct line of the answer sheet. Each question will be worth 1 point (12 points for the entire test) and no partial credit will be given. (Only the answer sheet will be turned in and graded.)
7. The overall team competition score will be calculated by adding the points for the team test (12 possible) to the points from the team participants individual percentage correct test scores (6 possible).
8. Hand-held calculators of any type are allowed. Internet access will not be allowed.

1. A boat is traveling on a river with a constant current. It takes the boat 1 hour and 50 minutes to travel 55 miles downstream (i.e. with the current) and 3 hours and 40 minutes for the return trip upstream (i.e. against the current). Find the speed of the boat (in mph) in still water.

2. Solve the inequality

$$|x^2 - 10| \leq 6.$$

3. Suppose the streets of a city are laid out in a grid, with streets running north-south and east-west. A police man patrols a 4×4 block grid starting in the center intersection. At each intersection he may elect to go north, south, east, or west. How many ways can he walk exactly 4 blocks and return to his starting place at the center?



4. A ball of volume $36\pi\,\mathrm{m}^3$ is inscribed inside a cube. What is the radius of a sphere that is circumscribed about the cube?

5. Find all solutions to the equation

$$\ln x + \ln(x-1) = 1.$$

6. Find all functions $f(n)$ whose domain and range are the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ that satisfy the equation

$$f(n) + f(n+1) + f(f(n)) = 3n + 1$$

for all natural numbers n in $\mathbb{N}$.

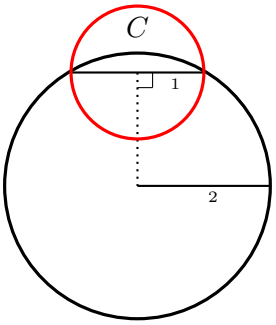
7. Write

$$\cos(\tan^{-1}(\sin(\cot^{-1} x)))$$

as an algebraic expression.

8. Terry was driving along an interstate highway at a constant speed using cruise control. She observed a mile marker containing a three-digit number in which the tens digit was zero. Exactly one hour after she passed the first mile marker she passed another mile marker containing a two-digit number which contained the first and last digits (not necessarily in that order) of the previously observed mile marker. Exactly one hour later she passed a third mile marker containing another two-digit number in which the digits were the same two digits that she observed an hour earlier. How fast is Terry going (in mph)?

9. Two circles, one of radius 1 and one of radius 2, intersect in such a way that the two points of intersection are diametrically opposite with respect to the smaller circle (see figure below). Find the area of the crescent-shaped region C (called a lune) lying outside the larger circle and inside the smaller circle.



10. Let $f(x) = \dfrac{x+1}{x+2}$. Find a formula for

$$f^7(x) = (\underbrace{f \circ \cdots \circ f}_{7 \text{ times}})(x).$$

(Note $f(x) = f^1(x)$).

11. A circle is inscribed in a 5-12-13 right triangle. What is its radius?

12. Three large cylindrical pipes are transported on a flatbed truck by laying the pipes flat, stacking one on top of the other two and using a tight metal band around all three to hold them together. If the outside diameter of each pipe is 3 feet, then what is the length of the metal band?

Team Competition Answer Sheet

School Name:
Question 1: mph
Question 2:
Question 3:
Question 4: m
Question 5:
Question 6:
Question 7:
Question 8: mph
Question 9:
Question 10:
Question 11:
Question 12: feet