The University of Alabama

Thirty-nineth Annual

High School Mathematics Tournament

Individual exam

December 5, 2020

1. Do **not** turn this page until the proctor indicates that it is time to begin.
2. The notation $(f \circ g)(x)$ refers to composition of functions: $(f \circ g)(x) = f(g(x))$.
3. Throughout the test, the letter i represents the imaginary unit $i = \sqrt{-1}$, $\log(x)$ means $\log_{10}(x)$, and $\ln(x)$ means $\log_e(x)$.
4. All answers must be exact, unless specifically asked to do otherwise. Leave π, e, and radicals in the answer.
5. The test is 2 hours in length.
6. Answers to the questions must be entered on the correct line of the answer sheet. Each question will be worth 1 point (18 points for the entire test) and no partial credit will be given. (Only the answer sheet will be turned in and graded.)
7. The overall team competition score will be calculated by adding the score (18 possible points) for each of the 6 team members together.
8. Hand-held calculators of any type are **not** allowed. Internet access or collaboration is not allowed.

1. Anthony, Brian, Chris, and David have four different colored hats that are red, green, blue, and black. If Anthony doesn't wear red, Brian doesn't wear green, Chris doesn't wear blue, and David doesn't wear black. How many different ways can they share the four hats?

2. Suppose that N is an integer and N^5 ends with the digit 7. What is the last digit of N itself?

3. Find the number of solutions (x, y) with x and y integers, to the system of equations

$$\begin{cases} y + |x| = 3 \\ |x|y + x^3 = 0 \end{cases}$$

4. When Dave walks to school, he averages 90 steps per minute, each of his steps are 75 cm long. It takes him 16 minutes to get to school. His brother, Jack, going to school by the same route, averages 100 steps per minute, but his steps are only 60 cm long. How long does it take Jack to get to school?

5. A marching band tries to form rows of 5 musicians each, but there are 4 musicians in the last row instead of 5. They then try to form rows of 7 musicians each, but there are 6 in the last row. Finally, they try to form rows of 11 musicians each, but there are 10 in the last row. There are less than 400 musicians in the band. How many are there?

6. Find exact expressions for all roots of $x^4 + 3x^3 - 13x^2 + 3x + 1$.

7. A cylindrical can is made up of a lateral surface that is metal square of side-length x and two circular disks. What is the volume of this can in terms of x.

8. Find the exact value of

$$\sin\left(\frac{7\pi}{48}\right)\cos\left(\frac{7\pi}{48}\right)\cos\left(\frac{\pi}{24}\right).$$

9. Sven builds 3-legged stools and 4-legged tables. Last month, Sven used 72 legs to build 3 more stools than he built tables. This month, he would like to build half as many stools as last month, but twice as many tables as last month. How many legs will he use this month?

10. Find the value of x such that

$$11^{102} = 11 + \sum_{k=0}^{100} \log_{10}\left[x^{(11^k)}\right]$$

11. Suppose that f is a function with the following special property: For all positive real numbers a and b, $f(a \cdot b) = f(a) + f(b)$. If $f(16) = 4$, what is $f(2)$?

12. Let a, b satisfy

$$a = b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cdots}}}, \text{ and } b = a - \cfrac{1}{b + \cfrac{1}{a - \cfrac{1}{b + \cdots}}}.$$

Find $(a^2 + b^2)^2$.

13. Someone made a mistake in building a clock: the minute hand moves like the hour hand is supposed to, and the hour hand moves like the minute hand is supposed to. Most of the time the hands are in positions that are not possible for a normal clock to take. Suppose the clock was showing the correct time at noon. What is the first time after noon that the hands are in a position that it is possible for a normal clock to have?

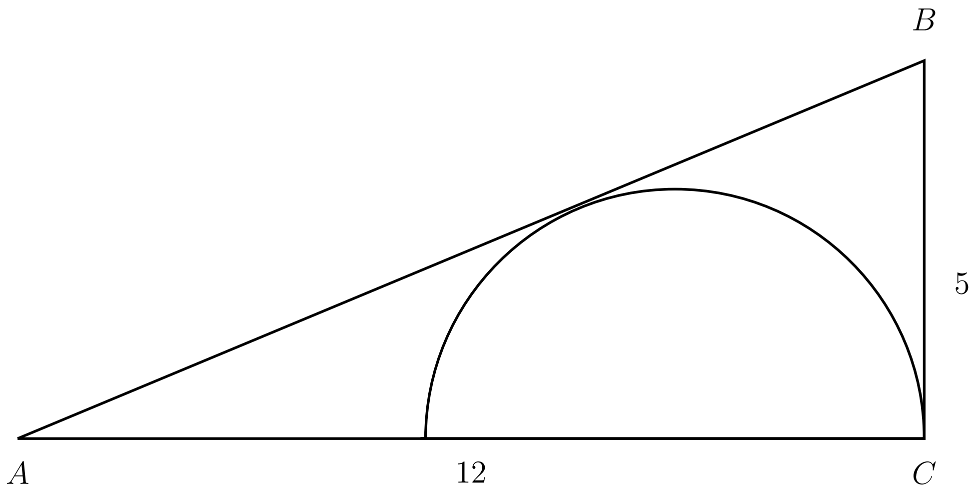
14. Let a, b, c satisfy $a + b + c = 2021$ and

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{2020}{2021}.$$

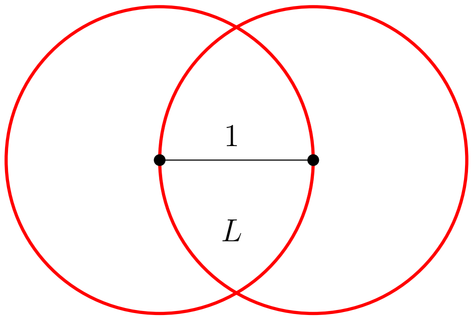
Find

$$\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a}.$$

15. In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown (the figure is not meant to be exact). What is the radius of the semicircle?



16. Suppose that two circles of radius 1 intersect in such a way that the centers of each circle lie on the other circle as shown below. Find the area of the 'lens' L composed of points that are contained on the inside intersection of both circles.



17. Let a, b, c, d be the roots of $f(x) = x^4 + 3x^3 + 3x + 2$. Find

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}.$$

18. Let the function $f(x)$ defined on the interval $(0, \infty)$ have the following properties

(a) $f(x) > 0$, for $x > 0$,

(b) $f(1) + f(3) = 20$

(c) $f(x + y) = f(x) + f(y) + 2\sqrt{f(x)f(y)}$ for any $x, y > 0$.

Find $f(2^{2020})$.

Answer Sheet

Student name, school, number:
Question 1:
Question 2:
Question 3:
Question 4:
Question 5:
Question 6:
Question 7:
Question 8:
Question 9:
Question 10:
Question 11:
Question 12:
Question 13:
Question 14:
Question 15:
Question 16:
Question 17:
Question 18: