The University of Alabama

Thirty-nineth Annual

High School Mathematics Tournament

Team Exam

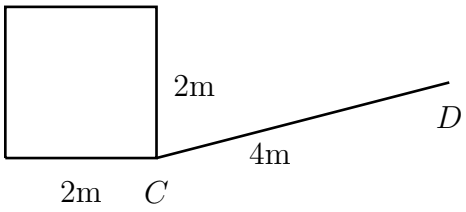
Novmeber 12, 2022

1. Do **not** turn this page until the proctor indicates that it is time to begin.
2. The notation $(f \circ g)(x)$ refers to composition of functions: $(f \circ g)(x) = f(g(x))$.
3. Throughout the test, the letter i represents the imaginary unit $i = \sqrt{-1}$, $\log(x)$ means $\log_{10}(x)$, and $\ln(x)$ means $\log_e(x)$.
4. All answers must be exact, unless specifically asked to do otherwise. Leave π, e, and radicals in the answer.
5. The test is 45 minutes in length.
6. Answers to the questions must be entered on the correct line of the answer sheet. Each question will be worth 1 point (18 points for the entire test) and no partial credit will be given. (Only the answer sheet will be turned in and graded.)
7. The overall team competition score will be calculated by adding the score (18 possible points) for each of the 6 team members together.
8. Hand-held calculators are allowed, but calculators with the ability to perform symbolic manipulation (i.e. CAS) are not allowed. Internet access will not be allowed.

1. Brian scored an average of 85 on his first four math tests. What is the lowest score that he would have to make on his fifth math test in order to raise his average to an 88?

2. Andrea has a bag of marbles. She gives 20% of them to her friend Ben. Then Andrea gives 10% of what is left to another friend, Carol. Finally, Andrea gives 25% of what is now left in the bag to her brother David. What percentage of her original bag of marbles does Andrea have left for herself?

3. A dog's leash is 4 m long and is attached to the corner of a 2 m $\times$ 2 m square doghouse at C, as shown. The dog is attached to the other end of the leash, at D. What is the area outside of the doghouse in which the dog can play?



4. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a 0.5 mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

5. Terry "solved" the equation $(x+\alpha)(-x+1011) = 2022$ by setting each of the factors on the left equal to 2022 and solving for x. Shockingly, he obtained the correct answers using this faulty reasoning. For what value of α does this occur?

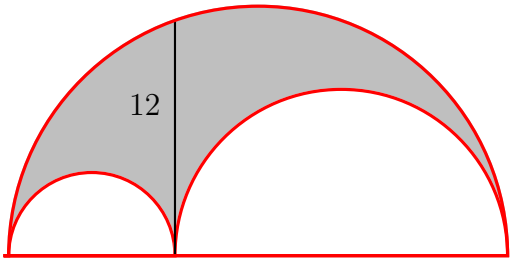
6. Suppose $f(x)$ is defined for all x and

$$f(x-3)+f(x+3)=0.$$

Suppose also that $f(x)=9-x^2$ for $-3\le x\le 3$. Find $f(1831)$.

7. Jeeves and Jeff have some change. Jeeves says to Jeff, "If you give me a nickel, then I'll have n times as much money as you." Jeff replies, "If you give me n cents, then I'll have 5 times as much money as you." If n is an integer, what is the largest possible value of n?

8. Two half-circles are inscribed in a larger half circle as shown below. A chord of length 12 runs perpendicular to the diameter of the largest half circle to a point on that circle. What is the shaded area?



9. Jack, Ken and Lee walk clockwise around a circular track. It takes Jack, Ken, and Lee respectively 6, 10, and 14 minutes to finish each lap. They start at the same time and at the same point on the track. How many minutes into the walk will they all be at the same spot on the track again?

10. Alice likes numbers whose digits strictly decrease and those that violate this condition in at most one place. In other words, if d_i is the ith digit, then $d_i \leq d_{i+1}$ for at most one value of i. For example, Alice likes $432100, 43210, 45210, 132$, and 3, but not the numbers 1337, 33221, and 123. How many 5 digit numbers does Alice like?

11. Let a be a number such that

$$\log_a(10) + \log_a(10^2) + \cdots + \log_a(10^{10}) = 110.$$

What is the value of a?

12. Real numbers a, b, c satisfy

$$a + ab + abc = 1,\ b + bc + bca = 3,\ c + ca + cab = 5.$$

Find the largest possible value of abc.

Team Competition Answer Sheet

School Name:
Question 1:
Question 2:
Question 3: m^2
Question 4: minutes
Question 5:
Question 6:
Question 7:
Question 8:
Question 9: minutes
Question 10:
Question 11:
Question 12: